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Shear band analysis and shear moduli calibration

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Abstract

Strain localization is a well known phenomenon, generally associated with plastic deformation and rupture in solids, especially in geomaterials. In this process, deformation is observed to concentrate in narrow zones called shear bands. This phenomenon has been studied extensively in the last 20 years by different researchers, experimentally, theoretically and numerically. A criterion for the onset of localization can be predicted solely on the basis of the constitutive law of the material, using the so-called shear band analysis. This criterion gives the critical orientation, and the critical stress state and strain for a given loading history. An important point, already stressed by Vardoulakis in 1980, is that in particular, *out-of-axes* shear moduli play a central role in the criterion. These are the moduli involved in the response to a deviatoric stress increment with principal axes oriented at 45° from total stress principal axes. Out-of-axes shear moduli are difficult parameters to calibrate; common tests, with fixed principal stress and strain directions, do not provide any information on these moduli, as long as they remain homogeneous. Still, real civil engineering and environmental problems are definitely not simple axisymmetric triaxial tests; practical modeling involves complex stress paths, and need complex parameters to be calibrated. Only special tests, like compression–torsion on hollow cylinder tests, or even more complex tests can be used for shear moduli calibration. However, shear band initiation in homogeneous, fixed-axes tests does activate out-of-axes shear. Hence, it is natural that shear band analysis makes shear moduli enter into the analysis.

Then, a typical inverse analysis approach can be used here: experimental observation of strain localization in triaxial tests can be used together with a proper shear band analysis for the model considered, in order to determine out-of-axes shear moduli.

This approach has been used for a stiff marl in the framework of a calibration study on a set of triaxial tests. The steps of the method are presented, and the bifurcation surface in the stress space is exhibited. © 2002 Published by Elsevier Science Ltd.

1. Introduction

During the last 20 years, a lot of work has been devoted to strain localization in solids, both on the experimental and theoretical sides. It is well known that strain localization is associated with rupture in many solids, from metals to geomaterials, including polymers, ceramics and other solids. As for soils and

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granular materials, experimental studies performed by Vardoulakis and co-workers (Han and Vardoulakis, 1991; Vardoulakis et al., 1978; Vardoulakis and Graf, 1985), Tatsuoka et al. (1990, 1986), Arthur and co-workers (Arthur and Dunstan, 1982; Arthur et al., 1977), Finno and co-workers (Finno et al., 1996; Finno and Viggiani, 1997), Desrues and co-workers (Bésuelle et al., 2000; Desrues, 1984, 1990; Desrues et al., 1996, 1985; Mokni and Desrues, 1999), and others (many other studies in rock) have established a number of conclusions, among which the following can be listed as motivation for the present discussion:

(1) *Strain localization in shear band mode can be observed in most*, *if not all*, *laboratory tests leading to rupture in geomaterials*, at least at sufficiently low temperature and pressure. The figure of the cylindrical triaxial specimen split into two blocks, after some barrel-shaped deformation, by a single shear plane is classical in soil mechanics (e.g. Fig. 1). However, shear banding is possible, and even very common, also in short specimens as illustrated in Fig. 2, and even in cubical triaxial devices with strain control using rigid platens, as shown in Fig. 3.

(2) *Complex localization patterns may be the result of specific geometrical or loading conditions*. In very short specimens, shear bands can reflect several times from the rigid boundaries of the specimen, as revealed on Fig. 4 by incremental strain field monitoring using stereophotogrammetry on a specially designed biaxial apparatus.

Looking at the second increment from the left on the figure, one can see that a quasi-complete shear band mechanism takes place suddenly at this time step (which is located just before the peak in the load–displacement curve). Whether such a localization pattern is the result of a propagation phenomenon, or it

Fig. 1. Classical triaxial test specimen with shear plane; *after J.L. Colliat-Dangus Doctoral Thesis*, *1984* (Colliat-Dangus, 1986).

Fig. 2. Short triaxial test specimen with shear plane(s), *Photo J. Desrues*.

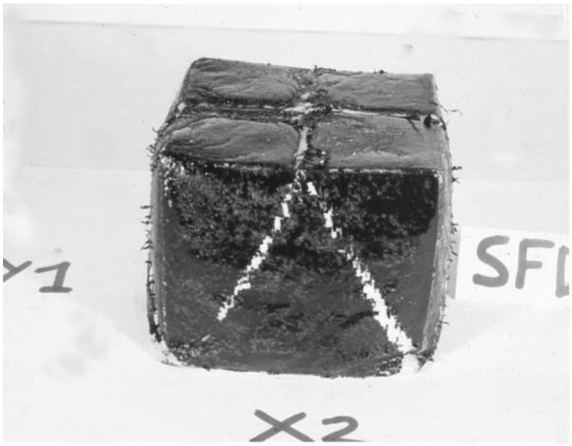

Fig. 3. Strain localization in a cubical specimen tested in a rigid-platen true triaxial apparatus (after Desrues et al., 1985).

rather corresponds to a "self-organization" process, is a matter for debate (see Desrues and Viggiani (in preparation) for details). Herein, it is proposed to consider this increment as the onset of localization, and the average orientation of the different branches of the band with respect to the vertical direction as the shear band orientation, predicted by the bifurcation criterion.

In axisymmetric tests, strain localization may remain more or less hidden inside the specimen, especially when improved test conditions are used, with reduced slenderness and refined anti-friction devices. However, Computed Tomography has made it possible to reveal that complex localization patterns can take place inside the specimen (Desrues et al., 1996); these patterns are a combination of plane strain

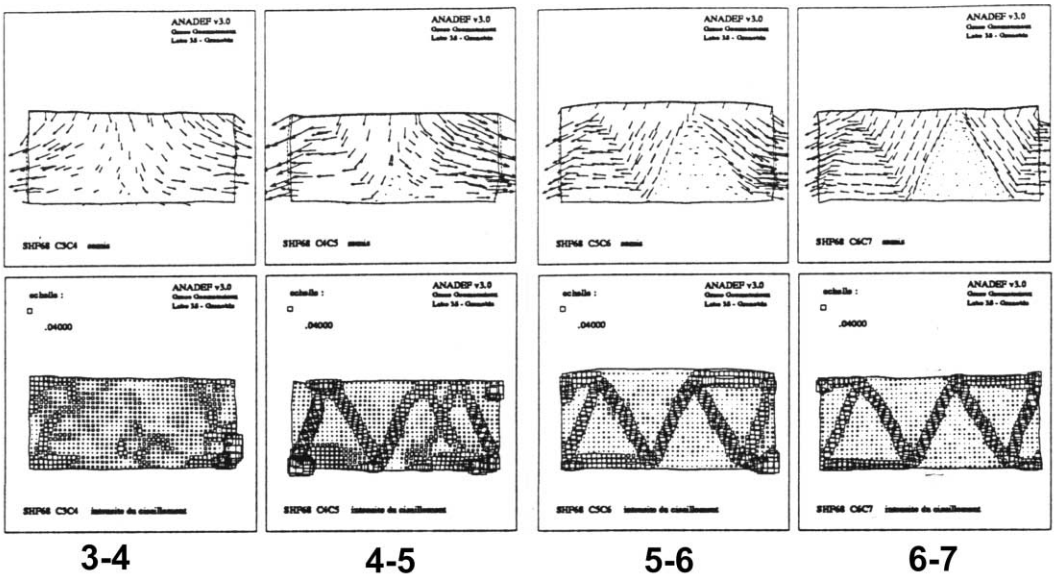

Fig. 4. Strain localization in a very short specimen tested in a plane strain biaxial apparatus. Four successive loading increments denoted $i-i+1$ are presented with i the serial number of the photograph. Photograph 4 is taken at the stress peak. The overall axial strain between two photographs is about 1.3%. One can see that an almost complete localization mechanism takes place within the deformation increment 4–5 (*after M. Mokni Doctoral Thesis* (Mokni, 1992)).

mechanisms with reflections on the top and bottom platens of the testing device, similar to those shown in cubical tests in Fig. 2.

(3) *Well marked stress peaks in stress–strain curves can be considered as the signature of an established shear band system over the specimens.* Plane strain experiments performed by Desrues et al. on sand, using stereophotogrammetry to detect precisely the onset of strain localization, have shown that the onset of strain localization is always detected at or before the peak in the axial stress versus axial strain curve (Desrues, 1984, 1998). Other experimental protocols have been used by different authors to detect the onset of localization, for example sets of extensometers placed on the outer surface of the specimen (Bésuelle, 1999; Bésuelle and Desrues, 2001; Goto et al., 1991), with the same result. It is sometimes difficult to conclude in the case of axisymmetric tests on ductile geomaterials (sand, normally consolidated clay, rocks under high confining pressure): when short specimens are used together with convenient end lubrication, triaxial axisymmetric tests may show only very smooth (if any) peaks. Computed tomography has made it possible to detect localization patterns in such tests, but the first onset of localization may have been missed, because the observation is not as direct as in biaxial tests with stereophotogrammetry. Hence, it is not always clear cut if the specimen has started to strain localize at peak or later. However, since in such cases the curve shows only a slow decrease of the deviatoric stress, then the stress state does not change much after peak. It follows that it can be considered more or less generally that a peak in the axial stress–strain curve indicates that the stress conditions for the onset of shear banding have been reached.

(4) *Simple measurements can be made to get an estimation of the orientation of the shear bands.* Direct observation after completion of the test, even measurements on good quality photographs later, can give precious information; in rock specimens, it has been shown by Bésuelle (Bésuelle, 1999; Bésuelle and Desrues, 2001) that the orientation can be measured quite accurately using the trace left on the inner surface of the membrane by the intense shearing and damage undergone by the material inside the band.

Fig. 5. Two clay specimens tested in axisymmetric triaxial conditions, one in compression (right) and the other in extension (left)—*Photo J. Desrues.*

(5) *Shear band orientation does not change drastically with Lode's angle.* Shear bands can be observed in triaxial compression tests, in biaxial tests, and in extension tests. Fig. 5 illustrates on a clay the two axisymmetric cases, compression (right) and extension (left): a single shear plane is clearly observed on the compression specimen, while a set of parallel ones, can be detected on the extension specimen. However, the orientation of the shear planes with respect to the major principal stress direction is comparable in the two cases (Note that the lateral stress is the major principal stress in the extension test; thus, the orientation of the shear band has to be measured with respect to the horizontal direction). Systematic measurements of the orientation of the shear bands for rectilinear tests along different directions in the deviatoric plane, i.e. different Lode's angle, have been performed by Arthur (Arthur and Dunstan, 1982) using his directional shear cell (DSC) apparatus and independently by Lanier et al. (Zitouni, 1988) using a True Triaxial Apparatus. These tests are called *constant b tests*, b being an alternative measure of the stress phase, defined by $b = (\sigma_I - \sigma_{II})/(\sigma_{II} - \sigma_{III})$. In both cases, a variation less than about 15–20° of the shear band orientation has been measured for Lode's angles from 0° (triaxial compression) to 60° (triaxial extension), which is small compared to the scatter in the experimental data. It is clear at least, that no discontinuity in shear band orientation can be expected on a continuous change of the stress path orientation in the deviatoric plane.

On the theoretical side, it has been established since the years 1970 by Rice and co-workers (Rice, 1976; Rudnicki and Rice, 1975), following previous work by Hadamard, Hill, Mandel, that the onset of shear banding in a semi-infinite homogeneous body subjected to a homogeneous loading history can be predicted solely on the basis of the constitutive equations of the solid. The subject of constitutive modeling has received considerable attention in the second part of the last century, especially in geomaterials; the basic concepts of Plasticity have given rise to a large number of variations, including non-normality, multi mechanisms, bounding surfaces, multilaminate models and other elasto-plastic extensions of the original framework; on the other hand, hypoplasticity has been introduced and developed in Europe (Chambon, 1989; Chambon et al., 1994a; Kolymbas, 1991) from a different point of view, not using the concepts of yield surface and plastic potential (see Tamagnini et al. (2000a) for a synthesis of hypoplasticity at the end of the 90s). Whatever the framework is, the evolution of constitutive equations toward a proper accounting of material behaviour complexity has led to incrementally non-linear constitutive equations (or equivalently rate-type non-linear constitutive equations). These developments have progressed from only two constitutive zones (loading/unloading with respect to a single mechanism), then to several constitutive cones in the stress space (related for example with several plastic mechanisms), and finally to thoroughly non-linear equations, for which any change in strain rate direction induces a change in the tangent constitutive response. As far as the localization criterion is concerned, incremental non-linearity introduces more complexity in the problem. Rice and Rudnicki (1980) have discussed in detail the case of mono-mechanism elastoplastic constitutive equations, introducing so-called continuous and discontinuous bifurcation for respectively loading in both inner and outer shear band, and loading inside/unloading outside situations. The framework of the linear comparison solid defined by Hill (1958) and extended by Raniecki and Bruhns (1981) to non-associated solids makes it possible to establish some theoretical results on the effect of the linearized approach (loading everywhere, i.e. continuous bifurcation) on the localization criterion. In the case of thoroughly non-linear constitutive equations, in general, the localization criterion becomes fully non-linear, which makes necessary some additional assumptions (e.g. Kolymbas, 1981) and/or numerical search for solutions. However, in the case of hypoplastic constitutive equations, it has been shown by Chambon and co-workers (Desrues and Chambon, 1989; Desrues et al., 1996) that due to the mathematical structure of the models, an explicit localization criterion can be derived analytically despite the thoroughly incremental non-linearity of the equations.

It follows from the theoretical studies briefly summarized above, that for most constitutive models for geomaterials a localization criterion can be checked and will predict possible localization on the basis of the pre-bifurcation material characteristics of the material. However, it is interesting to notice that among the constitutive moduli which enter in the criterion, so-called *out-of-axes* shear moduli play a major role. This was already observed by Vardoulakis in 1980 (Vardoulakis, 1980), who realized that this fact could be used in the purpose of parameter identification:

> "The interpretation of the experimental data in the light of the above bifurcation analysis ... yields an estimate of the incipient modulus μ."

In the following, the nature of the out-of-axes shear moduli and their relation with *in-axes* shear moduli is recalled, the different ways of characterizing experimentally these shear moduli is discussed, a strategy for a practical calibration of these moduli in relation with laboratory experiments is proposed, and illustrated with a real case.

2. Shear modes and shear moduli

2.1. Shear modes

Considering an elementary volume of material (the dashed square on top of Fig. 6) submitted to a given state of stress $\underline{\sigma}$, let us define two frames xy and XY respectively oriented parallel to and rotated by 45° with respect to the principal stress directions 1, 2. On the second row of the figure, two strain rate increments $\underline{\dot{\varepsilon}} = [\dot{\varepsilon}_{xx}, \dot{\varepsilon}_{yy}, \dot{\varepsilon}_{xy}]$ are defined in the xy frame, namely $\underline{\dot{\varepsilon}} = [a, -a, 0]$ and $\underline{\dot{\varepsilon}} = [0, 0, a]$, with a a positive scalar. The first one is a purely deviatoric strain rate, with its principal directions coincident with the axis xy; the second is purely deviatoric also, but its principal directions are coincident with axis XY. Row 3 in the figure illustrates the fact that, if a smaller square part of the body is considered, with its edges parallel to the XY frame, then for the same deformation rate the strain rate components expressed in the XY frame are simply exchanged with respect to the line 2. The strain rates (A) and (C) in the left column are two expressions of

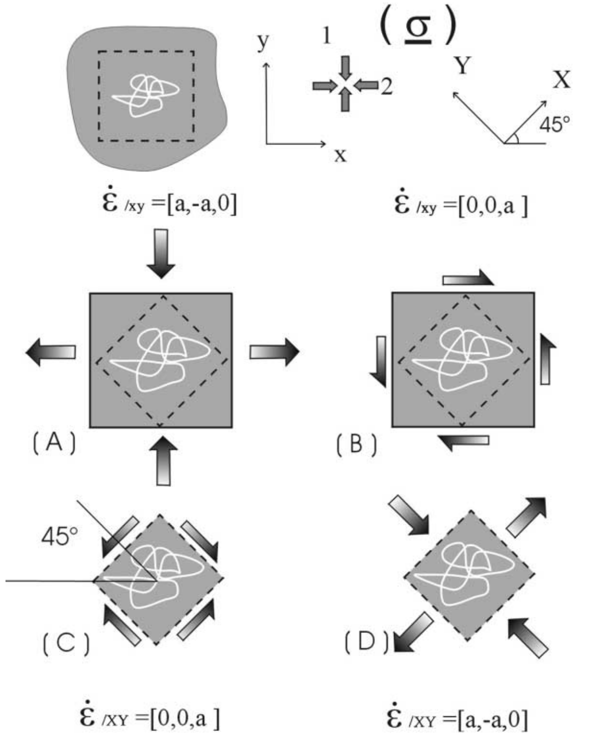

Fig. 6. Two different shear modes applied to an elementary volume: definition of in-axes and out-of-axes shear modes.

the same strain rate, *idem* (B) and (D); but (A) is different from (B), and (C) from (D). Let us call in-axis the shear modes (A) and (C), with reference to the principal stress axis; and out-of-axes the rates (B) and (D).

The strain rates (A)≡(C) and (D)≡(B) have the same invariants, only their principal directions are different. If the material behaves isotropically, all directions in the plane are equivalent with respect to material properties, then the two strain rates will produce the same stress rate response, only rotated accordingly. On the contrary, if the material is anisotropic incrementally, the stress rate response to in-axes and out-of-axes will be different, not only by a simple rotation.

2.2. Shear moduli

According to the previous discussion, the incremental stress–strain relations must account for a different response to the two different shear modes. This question has been studied by different authors. Biot (1939) has shown, in the framework of linearized elasticity that when an isotropic medium is under an initial (non-isotropic) stress, the incremental properties do not remain isotropic; in Biot (1963) he derives the incremental stress–strain relations for an incompressible material, showing that they contain two elastic coefficients N and Q corresponding to the shear modes called (A) and (B) here. Biot uses a finite strain framework, and he writes the incremental equations between the time derivative of Cauchy stress and the rate of the deformation gradient. Using an objective strain rate instead of time derivative of the Cauchy stress, e.g. Jaumann stress derivative, would extract the additional terms linked to initial stress from the two elastic coefficients N and Q, letting them equal. This remark does not contradict the result of Biot, but it is meant to help comparison with later works discussed here, in which the differences between the shear moduli does not come only from finite strain geometrical corrections. *In the sequel, all stress rates* $\dot{\sigma}$ *are supposed to be objective*. This essentially means that when using the constitutive equations in the context of finite strain calculations, e.g. finite strain finite element code, ad hoc corrections have to be added to transform the objective stress rate specified into the time derivative of the Cauchy stress (or other).

Hill and Hutchinson (1975) introduce the shear moduli μ and μ^* with the following definition:

$$\dot{\sigma}_1 - \dot{\sigma}_2 = \mu(\dot{\varepsilon}_1 - \dot{\varepsilon}_2) \tag{1}$$

$$\dot{\sigma}_{12} = \mu^* \dot{\varepsilon}_{12} \tag{2}$$

The same definition is used by Vardoulakis (1980). In the sequel we will call μ in-axes shear modulus and μ^* out-of-axes shear modulus. In conformity with tradition, one might prefer 2μ instead of just μ, however in the present paper we will use the same notation as the two references cited above.

Writing the incremental stress–strain relations (1) and (2) for the different shear modes considered in Fig. 6, we get for (A):

$$\dot{\sigma}_{xx} - \dot{\sigma}_{yy} = \mu(\dot{\varepsilon}_{xx} - \dot{\varepsilon}_{yy}) = 2\mu a \tag{3}$$

and for (B):

$$\dot{\sigma}_{xy} = \mu^* \dot{\varepsilon}_{xy} = \mu^* a \tag{4}$$

If the material is isotropic incrementally, stress–strain relations reads the same in xy and XY frame; then we have for (C):

$$\dot{\sigma}_{XY} = \mu^* \dot{\varepsilon}_{XY} = \mu^* a \tag{5}$$

Considering (A) and (C), which are two expressions of the same shear mode, one recognizes that $\dot{\sigma}_{XY}$ is the shear component on the direction at 45° from the xy frame, so we have:

$$\dot{\sigma}_{XY} = 1/2(\dot{\sigma}_{xx} - \dot{\sigma}_{yy}) \tag{6}$$

and then, using first (3) in (6) and recalling (5):

$$\dot{\sigma}_{XY} = \mu a = \mu^* a$$

which proves that in the incrementally isotropic case, $\mu = \mu^*$. The contrary is equivalent to saying that the material is incrementally *anisotropic*.

2.3. Shear moduli in different constitutive frameworks

It is interesting to consider what is the general response of the elastoplastic constitutive models which assume isotropic hardening. Because the yield function is defined in the principal stress space, any stress increment which does not change the invariants of the stress tensor, but changes the principal stress directions only, is neutral with respect to the yield criterion. Consequently, it does not induce any plastic strain, and the response is elastic. An out-of-axes shear increment corresponds exactly to this case: to the first order, the effect of adding such an increment to the actual stress state does not change the principal stress values, but the principal stress direction. It is easy to check this point by using a simple Mohr's circle construction. If the elastic part of the constitutive equation is isotropic, then the shear modulus is prescribed and constant. On the other hand, an in-axes shear increment will induce plastic strain, and produce a much softer response. As plastic strain progresses, along a monotonic loading history, the out-of-axes shear modulus becomes relatively stiffer with respect to the in-axes shear modulus.

Other elastoplastic models incorporate more advanced features like anisotropic hardening, or non-coaxiality; these models give more freedom to model a proper evolution of the out-of-axes shear modulus. Hypoplastic models offer the same flexibility.

2.4. Calibration of out-of-axes shear moduli

Out-of-axes shear moduli, once identified as independent variables needed in constitutive models, have to be calibrated, and this is a priori difficult. Indeed, in most laboratory tests, principal stress directions are fixed. Axisymmetric triaxial test, true triaxial tests on cubical specimens, biaxial test in plane strain are fixed-axes tests. Out-of-axes shear moduli are simply not activated in such tests. To calibrate these moduli, principal stress direction rotation is needed. Among laboratory tests relevant in that respect, some candidates fail in insuring satisfactory homogeneity over the specimen. For example all the common direct shear cells are, of course, of this type, because they impose shearing over a prescribed plane. Even the most refined Cambridge simple shear boxes (SSB) (Stroud, 1971), which by construction were not matching the shear stress reciprocity requirement on adjacent faces of the box, are not homogeneous. Arthur's DSC (Arthur, 1988) is a pioneering apparatus, which has produced very stimulating research data, but it is a prototype. The same is true for Grenoble's $1\gamma 2\varepsilon$ apparatus (Joer et al., 1992), which like Arthur's one makes possible general plane strain paths and proper boundary stress application. The hollow cylinder torsional test is probably the best candidate, but still not a common test.

Strain localization is however an opportunity to fill the experimental gap between fixed-axes tests and out-of-axes Shear moduli. Indeed, as mentioned before, these moduli enter explicitly in the bifurcation criterion which results from so-called *shear band analysis* (Chambon et al., 1994b; Vardoulakis, 1980). It is not surprising, because shear banding suddenly breaks the restriction on rotations imposed by the testing device and allows the specimen develop locally large strain, material rotation and principal stress and strain rate rotation. Hence out-of-axes moduli are expected to play a role in shear band analysis.

Coming back to several items of our list of motivation presented in introduction, let us recall that (i) shear bands are observable in most laboratory tests, and that (ii) the orientation can measured easily with reasonable accuracy. It is possible to use these observations to calibrate shear moduli.

3. CLoE hypoplastic model

CLoE model has been developed at the end of the 1980s. The model developed from the original heuristic one proposed by Chambon and Desrues in 1983. This early version successfully extended Rudnicki and Rice's shear band analysis, without any restrictive assumptions, to a class of thoroughly non-linear rate-type constitutive equations (Chambon and Desrues, 1985; Desrues, 1984; Desrues and Chambon, 1989). The name of the model is an acronym of the words consistency, localization and explicit. A detailed description of the model has been given in Chambon et al. (1994a). Only the main features are recalled hereafter.

3.1. Basic equations

CLoE model is a rate-type constitutive model, which gives the objective stress rate $\underline{\dot{\sigma}}$ as a tensorial non-linear function of the strain rate $\underline{\dot{\varepsilon}}$.

$$\underline{\dot{\sigma}} = \underline{f}(\underline{\dot{\varepsilon}}) \tag{7}$$

The mathematical expression of the constitutive equation is:

$$\underline{\dot{\sigma}} = \underline{\underline{\boldsymbol{A}}} : \underline{\dot{\varepsilon}} + \underline{\boldsymbol{b}}\|\underline{\dot{\varepsilon}}\| \tag{8}$$

with $\underline{\underline{\boldsymbol{A}}}$ and $\underline{\boldsymbol{b}}$ two constitutive tensors which depend on the evolution of state parameters. The non-linearity comes from the norm $\|\underline{\dot{\varepsilon}}\|$. Alternatively, one can write:

$$\underline{\dot{\sigma}} = \underline{\underline{A}} : (\underline{\dot{\varepsilon}} + \underline{b}' \|\underline{\dot{\varepsilon}}\|) \tag{9}$$

introducing $\underline{b}'$ which will be used in the sequel. CLoE shares this general form with its precursor, the heuristic model mentioned above (Chambon et al., 1994a), and also with the other models of the hypoplastic framework.

A few basic restrictions must be mentioned first. As for state parameters, the set of parameters considered in CLoE model is restricted to the stress state, meaning dependence on stress-state only, for sake of simplicity. This is a strong limitation because it does not make possible to model cyclic behaviour. Notwithstanding, unloading is properly modeled. Another restriction is that the material is initially isotropic.

An important feature of CLoE model is to use the concept of a limit surface defined in the principal stress space, which cannot be trespassed. The internal formulations of the law insure that no outer-directed stress rate can be generated for any stress state lying on the limit surface, and that this property is reached asymptotically when approaching the limit surface. This is one aspect of the consistency requirements automatically fulfilled in CLoE. Fig. 7 presents a cut of the limit surface in the deviatoric stress plane, and introduces the definition of the quantity $\bar{q}$, normalized deviator, which is the ratio of the deviatoric stress at the current stress point A, divided by the maximum admissible deviatoric stress at the limit stress state represented by point F (like failure) in the figure. The limit surface used is the van Eekelen surface (1980); it makes possible to fix independently the compression and extension friction angles.

The way in which the two constitutive tensors $\underline{\underline{A}}$ and $\underline{b}$ are defined differs a lot among Hyploplastic models. See Kolymbas (1991) and Tamagnini et al. (2000a) for an outline of the different classes of hypoplastic models. In CLoE model, $\underline{\underline{A}}$ and $\underline{b}$ can be written:

$$\underline{\underline{A}} = \begin{bmatrix} a & f' & e' & & & \\ f & b & d' & & & \\ e & d & c & & & \\ & & & g & & \\ & & & & h & \\ & & & & & j \end{bmatrix} \tag{10}$$

$$\underline{b}' = [k \quad l \quad m \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$$

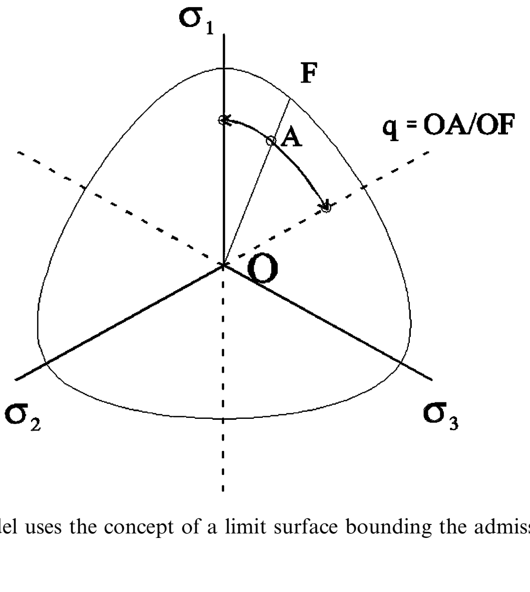

Fig. 7. CLoE model uses the concept of a limit surface bounding the admissible stress states.

for a general stress state, and

$$\underline{\underline{A}} = \begin{bmatrix} a & e' & e' & & & \\ e & b & d & & & \\ e & d & b & & & \\ & & & b-d & & \\ & & & & j & \\ & & & & & j \end{bmatrix} \tag{11}$$

$$\underline{b}' = [k \quad l \quad l \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$$

for an axisymmetric stress state. This is due to the choice of the stress state $\underline{\sigma}$ as the only state parameter. In this second expression, the number of independent moduli is much reduced with respect to the first one. The way CLoE model builds the general $\underline{\underline{A}}$ and $\underline{b}$ operators is based on an interpolation procedure between $\underline{\underline{A}}$ and $\underline{b}$ operators defined for axisymmetric stress paths, in compression and in extension. The reason is that axisymmetric tests are the most common laboratory tests for geomaterials, so that reliable stress–strain characteristics are commonly available on these paths.

Among the moduli appearing in Eq. (11), all those in the square sub-matrix at the upper left corner can be deduced from axisymmetric test data. As for the moduli on the diagonal, the first one is for shearing inside the axisymmetric plane 2–3, and it is linked to the moduli of the upper left square (according to the discussion of paragraph 2.2). However, the last two moduli called j cannot be deduced from axisymmetric experiments: they are precisely the out-of-axes shear moduli μ^* discussed in Section 2. Following the discussion of this previous section, we will use bifurcation criterion to calibrate these moduli.

Modulus j, being an out-of-axes shear modulus, has to satisfy the conclusions drawn in paragraph 2.2: if the material is isotropic, which in the case of CLoE is equivalent to saying that the stress state is isotropic, then out-of-axes shear moduli and in-axes shear moduli are equal. In such stress states, the upper part of the diagonal reduces to a common modulus a and the non-diagonal terms in the square to d. The in-axes shear moduli have to be $a - d$. This means that, for axisymmetric stress states, the three out-of-axes shear moduli on the lower part of the diagonal read $a - d$ also, like shown in Eq. (12).

$$\underline{\underline{A}} = \begin{bmatrix} a & d & d & & & \\ d & a & d & & & \\ d & d & a & & & \\ & & & a-d & & \\ & & & & a-d & \\ & & & & & a-d \end{bmatrix} \tag{12}$$

$$\underline{b}' = [k \quad k \quad k \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$$

This result indicates that the initial value of the out-of-axes shear modulus j is $j_0 = a - d$. As the stress state departs from the isotropic axis in a principal stress space, j departs from j_0. The evolution law assumed in CLoE is:

$$j = j_0(1 - \omega\bar{q}) \tag{13}$$

in which ω is a material parameter and $\bar{q}$ is the deviator stress normalized by the maximum deviator stress admissible for the same stress phase (or Lode's angle), as illustrated in Fig. 7.

Considering this evolution law, it is clear that the parameter to be calibrated in CLoE model is not j itself, but ω. Bifurcation analysis will give us the way to make the calibration.

3.2. Bifurcation criterion

The bifurcation criterion for CLoE results from a shear band analysis along the lines of Rudnicki and Rice (Rice, 1976; Rudnicki and Rice, 1975), except that the constitutive law is thoroughly non-linear, and this need some development to arrive at the explicit analytical criterion. Only the result is given here, details can be found in Chambon et al. (2000). Another useful reference is Tamagnini et al. (2000b) in which a comparison is given between the different hypoplastic laws with respect to shear band analysis, showing that the same general approach can be used for all of them. As for CLoE, the general criterion reads:

$$\|C\| = \|\tfrac{1}{2}(P_{il}^{-1} b_{lk} n_k n_j + P_{jl}^{-1} b_{lk} n_k n_i)\| \geqslant 1 \tag{14}$$

with

$$P_{ik} = \mathscr{M}_{ijkl} n_l n_j$$

and

$$\mathscr{M}_{ijkl} = A_{ijkl} + 1/2(\sigma_{il}\delta_{jk} - \sigma_{ik}\delta_{jl} + \sigma_{jl}\delta_{ik} - \sigma_{jk}\delta_{il})$$

Once expressed for a given set of parameters and for a given stress state, the criterion (14) becomes a polynomial expression of degree 4 in $\tan^2(\theta)$:

$$\|C\| - 1 = \Lambda_0 + \Lambda_1 \tan^2(\theta) + \Lambda_2 \tan^4(\theta) + \Lambda_3 \tan^6(\theta) + \Lambda_4 \tan^8(\theta) \geqslant 0 \tag{15}$$

where Λ_i are functions of the components of the constitutive tensors $\underline{\underline{\boldsymbol{A}}}$ and $\underline{\boldsymbol{b}}'$. Among those components, j is the only one which has not been already calibrated using the experimental data from triaxial axisymmetric tests. More precisely, the free parameter is ω defined in Eq. (13).

Fig. 8 presents a plot of the criterion as a function of $\tan^2(\theta)$ for increasing values of the loading parameter during the integration of the constitutive law for an elementary volume along a triaxial stress path.

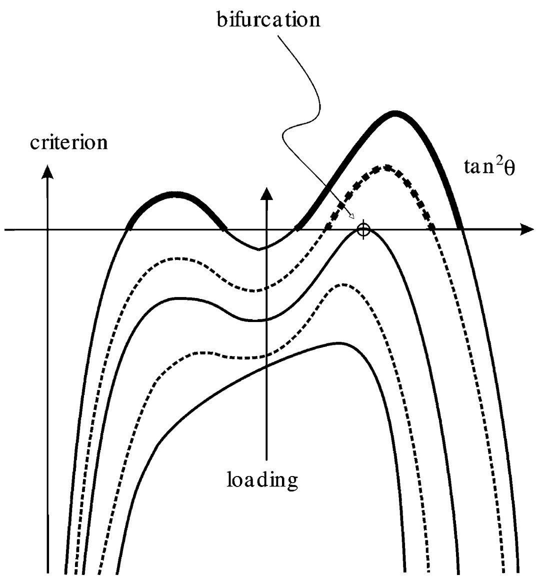

Fig. 8. The bifurcation criterion for CLoE model is a complete fourth degree polynomial function in $\tan^2(\theta)$. It may have one or two maximum, depending on its coefficients. During an integration of the model along a given path, the shape of the criterion changes. The bifurcation condition is met as soon as one root exists for which the criterion is null.

The criterion may have one or two maxima, depending on the coefficients Λ_i. These coefficients change as the loading progresses. The two maximum have no reason to be equal. At the beginning of the loading, the criterion is negative everywhere. The first bifurcation is encountered when the upper maximum touches the horizontal axis, i.e. for the first positive root in $\tan^2(\theta)$. This root defines two symmetric possible orientation θ and $-\theta$. If loading progresses, the maximum crosses the axis, so there is a range of orientations possible. It may happen that the second maximum comes to touch and then cross the axis also; then a second range of possible orientations is obtained. Only the first event is interesting on a loading program, namely when the first maximum touches the axis; this will be considered as the bifurcation point.

It is worth noticing also that if the stress path considered for the loading program changes, the shape of the criterion changes also because the moduli of the incremental law depend on the stress path followed. Considering for example rectilinear stress paths in the deviatoric plane, each having a constant Lode's angle, it may happen that the two maxima become equal and exchange their role of upper and lower maximum for a given Lode's angle. A discontinuity in the orientation predicted for the first bifurcation will occur in that case, which is not acceptable with respect to experimental observations presented in the introduction. This point is discussed later.

4. Parameter identification strategy

CLoE model is not only a theoretical development; it can be used in finite element codes. A calibration procedure has been defined for the model, and programmed as a computer code. It is not the purpose of the present paper to enter into the details of the procedure; they can be found in Chambon et al. (1994a). It is enough to say here that on the basis of experimental stress–strain curves recorded on different stress paths (axisymmetric triaxial compression, extension, isotropic tests), a sequence of interactive steps makes possible to define by iteration a set of material parameters which best reflect the user's interpretation of the data.

Indeed, using experimental data to calibrate a model needs always some interpretation to be made. One has to decide what is relevant in the data, and what should be left aside, because the real test conditions does not match anymore the requirements of the element test for which the mathematical formulations have been written. In other words, the significance of the characteristic curves recorded experimentally with respect to a given theoretical framework has to be checked by the user. For example, the significance of a peak in stress–strain curves is an important point, especially if a bifurcation criterion is to be used in the analysis.

4.1. Peak stress as bifurcation point

Indeed, peaks in experimental curves can be interpreted in different ways. One possible way is softening: one can choose to assume that the specimen remains homogeneous after the peak, and elaborate constitutive formulations which incorporate a progressive reduction of material strength, due to some homogeneous material degradation. This way has some drawbacks, among which the fact that it relies on experiments most likely affected by localization to establish a supposed homogeneous softening law.

Another way is to consider that whatever happens after a peak in a laboratory test is the response of a structure, not an elementary material response. This is obvious when the specimen falls into pieces just after the peak. In more ductile tests, the experimental findings recalled in the introduction show that the first interpretation is still the most relevant in a large number of cases, if not all. Moreover, the second leads to ill-posed problems in analytical and numerical simulations.

The modeling choice made with CLoE is to consider that the peak is the signature of the onset of strain localization in the specimen. Due to localization, the stress state evolution toward the limit surface in a triaxial test is interrupted. Highly concentrated material degradation takes place inside the shear band, inducing *local* softening. Experimental evidence has been shown in Desrues et al. (1996) that so-called *Critical State void ratio* in sand dense specimens is reached only inside the shear bands, and should not be confused with the more or less arbitrary plateau shown by volumetric strain curves as the result of global measures of the volume change of the localized specimens.

Consequently, no attempt should be made to adjust the peak stress value recorded in the tests with the limit surface. On the contrary, attention should be focused on properly modeling the pre-peak stress–strain response recorded, and then adjust the out-of-axes shear moduli to make the bifurcation criterion to be met at the peak.

Fig. 9 illustrates these dispositions: a specimen of a stiff marl has been tested in axisymetric triaxial compression; experimental (thick lines) and theoretical (thin lines) stress–strain responses are shown in the figure. Upper curves are axial stress–strain curves, lower curves are volumetric versus axial strain. The proposed interpretation of the experimental curves is the following: the specimen shows continuous contractancy up to the peak in stress–strain curves; then localization is initiated, gross fracturing develops quickly because the material is stiff and fine grained. As a result, global strength drops and dilatancy develops due to the interplay of the fractures. Theoretical curves are adjusted to fit the pre-peak part of the experimental curves; no attention is paid to the possibly unrealistic asymptotic strength and contractancy predicted far after the peak: the model is not intended to be used in calculations after the bifurcation criterion is met. It is interesting to notice that this choice makes possible to reach rupture load (i.e. bifurcation loads) with still stiff tangent moduli, which is consistent with experimental observations in laboratory tests.

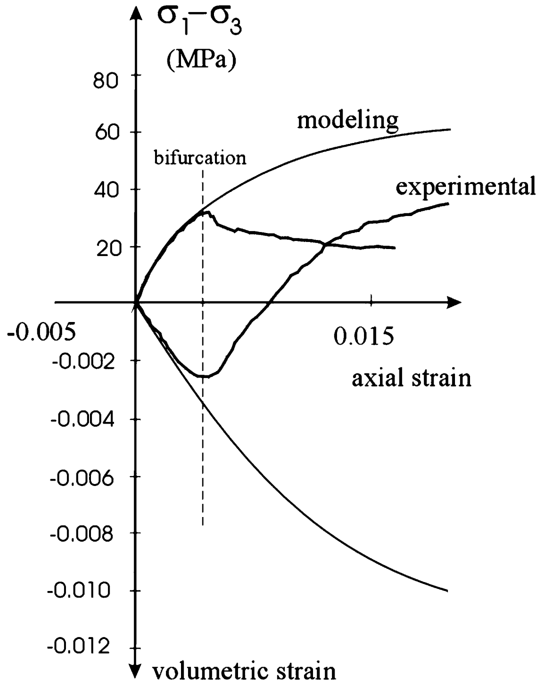

Fig. 9. Experimental and theoretical responses of a specimen tested on a axisymmetric triaxial path. As soon as homogeneity of the specimen is lost experimentally, the theoretical prediction should not be used further. The localization criterion must be adjusted in order to coincide with experimentally observed loss of homogeneity. The post-peak part of the formulation is not used.

4.2. Localization criterion used to calibrate shear moduli

The next and final step in the proposed strategy is to calibrate the out-of-axes moduli using localization criterion. The criterion is given by Eq. (15), in which the Λ_i are functions of the different moduli. Eq. (13) defines the formulation of j modulus, which depends on the ω parameter. Given an experimental stress path, with a peak stress occurring at a stress state σ_p, and an axial strain ε_p, a parametric study of the explicit criterion can be performed by integrating the model along this loading path for ω in a range between 0 and 2, e.g. If ω excesses 1.0 clearly j will become negative at the limit surface, which is not acceptable; however, since the constitutive model is not intended to be used beyond the bifurcation limit, the only requirement on ω is that j must be non-negative before and at bifurcation. The desired value for ω is obtained as the one for which the criterion is met at the same time as the experimental peak $(\sigma_p, \varepsilon_p)$. The criterion gives also the orientation of the shear band; if the same ω value does not provide a good prediction of both the peak stress–strain and the orientation, a compromise has to be found. If the discrepancy is too big, changes in the previous steps of the calibration should be considered.

Fig. 10 shows the evolution of the axial strain ε_p (left) and shear band orientation θ (right) at localization for the test presented in Fig. 9. The target values are $\varepsilon_p = 0.0043$ and $\theta = 77°$. With $\omega = 1.4$, one gets respectively 0.0044 and 80°5 for ε_p and θ, which can be considered in reasonable agreement if the experimental uncertainties are taken into account.

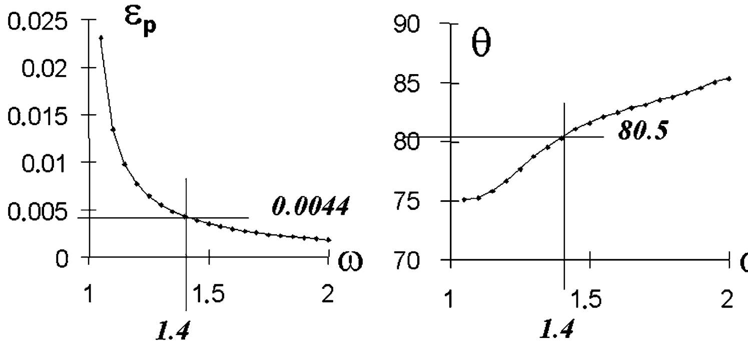

Fig. 10. Calibration of out-of-axes shear modulus parameter ω using a parametric study of the dependence of the critical axial strain ε_p and shear band orientation θ on ω.

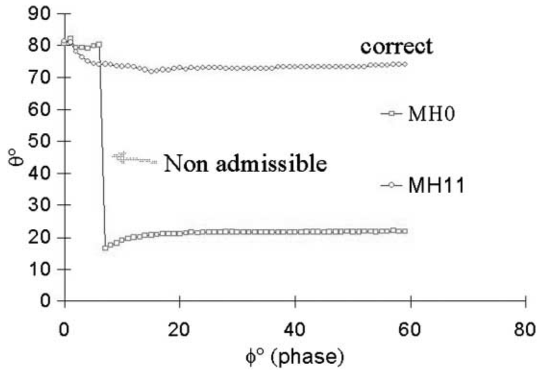

Fig. 11. Shear band orientation evolution with Lode's angle can become discontinuous in certain cases. This has to be checked for correct parameter calibration.

However, it may happen that a unique constant value of ω be insufficient for a consistent prediction of localization over the entire range of possible stress paths. At the end of the discussion of the bifurcation criterion in paragraph 3.2, it was mentioned that in some cases, a change of the first maximum touching the horizontal axis in Fig. 8 will induce a discontinuity in the evolution of the shear band orientation with respect to Lode's angle in the deviatoric plane. This is in contradiction with one of the experimental conclusions recalled in the introduction: to the author' knowledge, nothing like a jump in shear band orientation for a given b value, in constant b test programs, has been reported up to now in geomaterials. On the contrary, smooth variations have been observed by several authors (cf. references cited in the introduction).

An example of such pathological behaviour of the shear band orientation prediction for a complete scanning of the deviatoric plane between the extreme Lode's angles 0° (triaxial compression) and 60° (triaxial extension) is given in Fig. 11. Two set of parameters are shown, one giving a discontinuity around a

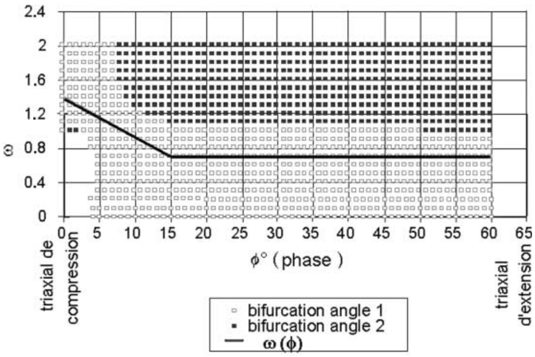

Fig. 12. A map of the occurrence of left or right maximum of the criterion as the first solution for bifurcation in a plane (ω, Lode's angle) can be used to establish a profile $\omega(\phi)$ which provides a continuous orientation evolution with ϕ.

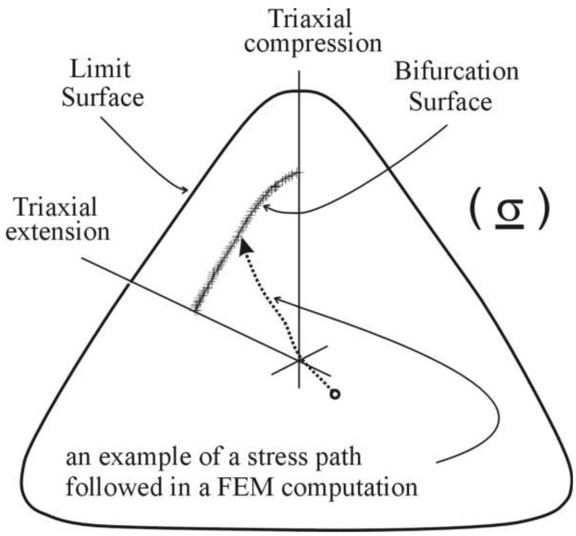

Fig. 13. The bifurcation surface of the model can be exhibited by a systematic exploration of the deviatoric plane, integrating the model with bifurcation check along rectilinear so-called b-constant path in the deviatoric plane.

Lode's angle of 7° approximately, and the other not. The correct response has been obtained by introducing a dependency of the parameter ω on the Lode's angle. This enhancement of the model is illustrated in Fig. 12, where a mapping of the space (ω, ϕ) is presented. In this mapping, the symbols have the following meaning: open symbol, first bifurcation is given by the maximum "1" of the criterion; filled symbol, maximum "2" comes first. For a smooth variation of the shear band orientation, any exchange of the order of the maximum must be rejected. The simplest profile for ω variation with ϕ is chosen: a piecewise linear evolution represented on Fig. 12. This corresponds to the second set of parameters, which gives the non-discontinuous response in Fig. 11.

Finally, Fig. 13 shows the set of critical stress points in the deviatoric plane of the stress space, obtained by the systematic exploration of the response of the model on rectilinear stress paths at constant Lode's angle in the deviatoric plane. This set of critical points defines a *Bifurcation Surface*, which is well inside the Limit Surface and also different in shape. Only one sixth of the surface is represented, because of symmetry.

5. Discussion

One may wonder if using the localization criterion to determine some parameters of the model is not simply adjusting to fit experimental data available. Moreover, the need of additional variation of ω with Lode angle to avoid discontinuous change in shear band orientation may give the impression that the model predicts only the response of tests used to fit it. It is the authors' contention that this is not the case. CLoE model is able to predicts the stress rate response $\underline{\dot{\sigma}}$ to any strain rate $\underline{\dot{\epsilon}}$ for any stress state $\underline{\sigma}$, on the only basis of axisymmetric triaxial stress strain characteristics in compression and extension, plus isotropic test response, and some limited additional input concerning unloading branches on these paths. As discussed in the paper, out-of-axes shear moduli are not constrained by such an experimental basis: they are simply not activated on these paths. There is a need for a way to calibrate these moduli with respect to experimental data. The bifurcation criterion used with shear band observations is a much simpler solution than performing hollow cylinder torsional tests. Of course, if the purpose of this work was only to compare the predictions of the criterion with experimental shear band data, then it would be a useless exercise. However, the model is intended to be used in finite element or other numerical codes. In this context, principal stress rotation will occur virtually anywhere in the domain, and out-of-axes moduli will be activated. Moreover, the bifurcation criterion will be checked in every stress point and the onset of localization will be predicted, along with shear band orientation. Proper post-bifurcation modeling imposes continuity between pre- and post-bifurcation model responses at bifurcation. The change from the first to the second kind of model, whatever they are, has to take place at the bifurcation surface of the pre-bifurcation model (see Chambon and Crochepeyre (1998) for details). This transition cannot be arbitrary defined, for example by an ad hoc surface in the stress space, which would be fitted on experimental peak stress states, independently from any bifurcation criterion. Instead, it is essential to calibrate the parameters of the model in order to met bifurcation criterion at the experimental peak load.

Of course, in hollow cylinder tests one is able to isolate the effect of the out-of-axes shear moduli. If such data were available, the bifurcation criterion could be compared with experimental shear band observations directly. In case of discrepancies, unknown factors other than out-of-axes shear moduli should be researched, identified, possibly incorporated in the model formulation. This would certainly improve the knowledge of material mechanical properties. However, in the present state of engineering, compression-torsion hollow cylinder tests are seldom available, especially for geomaterials. A model requiring hollow cylinder tests for calibrating its parameters would be of little help for practical applications.

Another point to discuss is the significance of out-of-axes shear moduli on localization, depending on the pre-bifurcation deformation state. For those states that contain a plane of zero-extension, e.g. plane strain, shear bifurcation can occur without any, or with limited change in the strain rate direction. It is possible that the rate of the deformation gradient inside the band is simply proportional to the rate outside, leading to no principal axes rotation. In this special case the out-of-axes shear moduli does not play any role. However, this is a restriction to the general plane strain deformation, thus not likely the most critical mode. On the other hand, the difference between axisymmetric strain rate and plane strain rate is the existence of a zero-deformation direction in the second case. The transition from one to the other, which must occur when localization takes place, does not necessarily involve more (or less) principal strain and stress rate direction rotation, than localization occurring in pre-bifurcation plane strain states. Thus, the significance of out-of-axes shear moduli on localization seems not to depend much on the pre-bifurcation deformation state.

A last remark should be made to compare the model discussed here with other models able to incorporate softer out-of-axes shear moduli. Amongst other are the so-called deformation theory of plasticity (Budiansky, 1959), and the vertex models (Rudnicki and Rice, 1975). These models share some important properties with respect to shear band modeling.

A shear band analysis can be seen as a non-uniqueness study which alternative solutions involve a shear band. Very often the additional hypothesis assuming that the fundamental solution and the alternative one can be described by the same linear formulae giving the stress rate as a function of the velocity gradient (usually corresponding to the loading behavior of the studied material). This point is discussed by Rice (1976) who defined continuous bifurcations as the ones obeying the previous assumption. He defined discontinuous bifurcation as the ones which cannot be studied with this assumption. As there is some experimental evidence that post localization is accompanied by unloading outside the shear band, the study of localization outside the restricted case of continuous bifurcation in the Rice' sense seems very important. On the other hand, it has been proven early that for very classical models, the continuous bifurcation assumption is sound (Rice and Rudnicki, 1980), but this result is not general (see the recent discussion done by Chambon et al. (2000) and the papers quoted in this discussion).

Deformation theory of plasticity and vertex model are models which prediction are only available for proportional or near proportional loading. The only possible study for such models is then the study of continuous bifurcation, as a complete study (i.e. incorporating the study of discontinuous bifurcation) cannot be performed. On the contrary, hypoplastic models like CLoE are generally able to predict behavior along unloading as well as loading stress paths. They are complete. Moreover a closed form general bifurcation criterion can be established which makes useless a restricted continuous bifurcation analysis, and it has been proven early for a very simple version of the model that the continuous bifurcation happens later in the loading process (Chambon and Desrues, 1985), illustrating the clear advantage of the possibility of a general shear band analysis.

6. Conclusion

Strain localization observed in laboratory tests is not an artificial effect due to "imperfect" test conditions and practice; it is an essential aspect of rupture, because rupture leads to localized deformation in most cases, both in the laboratory and in the field. Moreover, the theoretical bifurcation approach called shear band analysis, used with experimental observation of the onset of shear banding, gives a unique opportunity to obtain information on the so-called out-of-axes shear moduli, which otherwise require more difficult tests to be performed. The evolution of constitutive equations during the last 30 years have given rise to a number of advanced models, whose complexity makes possible nowadays non-trivial bifurcation predictions. CLoE model, a member of the hypoplastic family, has been used to illustrate these ideas. The proposed approach for shear moduli calibration has proven its efficiency. It should be recalled in conclusion that the incidence of a proper calibration of shear moduli on the performances of computer simulations of engineering problems is not only concerning localization prediction, which would be already very important, but also the results obtained before localization, since most models used in computer simulations nowadays do not allow to confuse in-axes and out-of-axes incremental shear moduli.

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